Faktorijele

Što je to?

- Umnožak prvih n prirodnih brojeva
- Označavamo sa $n!$
- Primjerice:

$$1! = 1$$
$$2! = 1 \cdot 2$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$$
$$27! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 25 \cdot 26 \cdot 27$$

- Dakle, za prvih n prirodnih brojeva vrijedi:

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-2) \cdot (n-1) \cdot n$$

Što je s nulom???

- Definiramo:

$$0! = 1$$

- Faktorijele zadovoljavaju rekurzivnu formulu, tj.

$$n! = n \cdot (n-1)!$$

- Pogledajmo kako to izgleda na primjeru 7!

$$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$$

No, znamo da je $6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$.

Ako malo pogledamo, možemo uočiti da zaista vrijedi rekurzivna formula jer u našem slučaju (ali i svim ostalim) je:

$$7! = 7 \cdot 6!$$

- Probajmo sada napisati rekurzivnu formulu za sljedeće faktorijele:

$$78!, 153!, 9!, 13!$$

$$78! = 78 \cdot 77!$$
$$153! = 153 \cdot 152!$$
$$9! = 9 \cdot 8!$$
$$13! = 13 \cdot 12!$$

Idemo sada na zadatke

1. Izračunaj:

$a)\ \frac{25!}{20!}$

$b)\ 7! + 8! + 9!$

$c)\ \frac{7!-6!}{120}$

Rješenje:

$a)\ \frac{25!}{20!} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 20} = 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25 = 6\ 375\ 600$

Može i ovako:

$$\frac{25!}{20!} = \frac{20! \cdot 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25}{20!} = 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25 = 6\ 375\ 600$$

$b)\ 7! + 8! + 9! = 7! + 8 \cdot 7! + 9 \cdot 8 \cdot 7! = 7! \cdot (1 + 8 + 9 \cdot 8) = 7! \cdot 81 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 81 = 408\ 240$

$c)\ \frac{7! - 6!}{120} = \frac{7 \cdot 6! - 6!}{120} = \frac{6! \cdot (7-1)}{120} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 6}{120} = \frac{120 \cdot 6 \cdot 6}{120} = 6 \cdot 6 = 36$

2. Zapiši kraće:

$a)$ $15 \cdot 14 \cdot 13 \cdot 12! = 15!$

$b)$ $105 \cdot 104 \cdot 103! = 105!$

$c)$ $(n-3) \cdot (n-2) \cdot (n-1)! = (n-3)!$

$d)$ $(n+2) \cdot (n+1) \cdot n \cdot (n-1)! = (n+2)!$

3. Skrati razlomke:

a) $\frac{13!}{11!} = \frac{13 \cdot 12 \cdot 11!}{11!} = 13 \cdot 12 = 156$

b) $\frac{54!}{53!} = \frac{54 \cdot 53!}{53!} = 54$

c) $\frac{n!}{(n-2)!} = \frac{n \cdot (n-1) \cdot (n-2)!}{(n-2)!} = n \cdot (n-1)$

d) $\frac{(n+1)!}{(n-2)!} = \frac{(n+1) \cdot n \cdot (n-1) \cdot (n-2)!}{(n-2)!} = (n+1) \cdot n \cdot (n-1)$

e) $\frac{2n(2n-1)}{(2n)!} = \frac{2n(2n-1)}{2n \cdot (2n-1) \cdot (2n-2)!} = \frac{1}{(2n-2)!}$

4. Izračunaj:

a) $\frac{5!+4!}{3!} = \frac{5\cdot4\cdot3!+4\cdot3!}{3!} = \frac{3!\cdot(5\cdot4+4)}{3!} = 20 + 4 = 24$

b) $\frac{50!}{48!} - \frac{30!}{28!} = \frac{50\cdot49\cdot48!}{48!} - \frac{30\cdot29\cdot28!}{28!} = 50 \cdot 49 - 30 \cdot 29 = 1\,580$

c) $\frac{40!}{39!} + \frac{39!}{38!} + \cdots + \frac{2!}{1!} + \frac{1!}{0!} = \frac{40\cdot39!}{39!} + \frac{39\cdot38!}{38!} + \cdots + \frac{2\cdot1!}{1!} + 1 = 40 + 39 + \cdots + 2 + 1 = \frac{40(40+1)}{2} = 820$

d) $\frac{1}{(n-1)!} - \frac{1}{n!} = \frac{n!-(n-1)!}{n!(n-1)!} = \frac{n(n-1)!-(n-1)!}{n!(n-1)!} = \frac{(n-1)!(n-1)}{n!(n-1)!} = \frac{n-1}{n!}$

e) $\frac{(n-1)!}{n!} + \frac{(n-2)!}{(n-3)!} = \frac{(n-1)!}{n(n-1)!} + \frac{(n-2)(n-3)!}{(n-3)!} = \frac{1}{n} + (n-2) = \frac{1+n(n-2)}{n} = \frac{1+n^2-2n}{n} = \frac{(n-1)^2}{n}$

5. Riješi jednadžbe:

a) $\frac{(n+2)!}{n!} = 72$

b) $\frac{k!}{(k-4)!} = \frac{2k!}{(k-2)!}$

c) $\frac{n!-(n-1)!}{(n+1)!} = \frac{1}{6}$

Rješenja:

a) $\frac{(n+2)!}{n!} = 72$

$$\frac{(n+2)\cdot(n+1)\cdot n!}{n!} = 72$$

$$(n+2)(n+1) = 72$$

$$n^2 + n + 2n + 2 - 72 = 0$$

$$n^2 + 3n - 70 = 0$$

$$n_{1,2} = \frac{-3 \pm \sqrt{9 - 4\cdot 1\cdot(-70)}}{2} = \frac{-3 \pm 17}{2}$$

$$n_1 = 7,\ \ n_2 = -10$$

Rješenje koje zadovoljava je prirodan broj, tj. $\mathrm{n} = 7$

b) $\frac{k!}{(k-4)!} = \frac{2k!}{(k-2)!}$

$$\frac{k(k-1)(k-2)(k-3)(k-4)!}{(k-4)!} = \frac{2k(k-1)(k-2)!}{(k-2)!}$$

$$k(k-1)(k-2)(k-3) = 2k(k-1)$$

$$(k-2)(k-3) = 2$$

$$k^2 - 3k - 2k + 6 - 2 = 0$$

$$k^2 - 5k + 4 = 0$$

$$k_{1,2} = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2} \Rightarrow k_1 = 4, \qquad k_2 = 1$$

c) $\frac{n!-(n-1)!}{(n+1)!}=\frac{1}{6}$

$$\frac{n(n-1)!-(n-1)!}{(n+1)n(n-1)!}=\frac{1}{6}$$

$$\frac{(n-1)!\,(n-1)}{(n+1)n(n-1)!}=\frac{1}{6}$$

$$\frac{n-1}{n(n+1)}=\frac{1}{6}$$

$$6(n-1)=n^2+n$$

$$6n-6-n^2-n=0$$

$$-n^2+5n-6=0$$

$$n^2-5n+6=0$$

$$n_{1,2}=\frac{5\pm\sqrt{25-24}}{2}=\frac{5\pm 1}{2}\Rightarrow n_1=3, n_2=2$$

Binomni koeficijent

$$\binom{n}{k} = \frac{n(n-1)(n-2)(n-3)\ldots(n-k+1)}{1\cdot 2\cdot 3\ldots\cdot k}$$

Pri čemu je n prirodan broj, k prirodan broj ili 0 te $k \leq n$

Možemo zapisati pomoću faktorijela

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

- Vrijedi svojstvo simetrije

$$\binom{n}{k} = \binom{n}{n-k}$$

1. Izračunaj:

a) $\binom{9}{4} = \frac{9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4} = 126$

Možemo i ovako:

$$\binom{9}{4} = \frac{9!}{4!\cdot5!} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}{1\cdot2\cdot3\cdot4\cdot1\cdot2\cdot3\cdot4\cdot5} = 126$$

No, primijetimo da je u ovakvom slučaju elegantiniji i brži prvi način pa ćemo tako riješiti naredne primjere.

b) $\binom{15}{7} = \frac{15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} = 6\,435$

$c)\ \binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} = 1\,140$

$d)\ \binom{7}{4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = 35$

2. Računom provjeri jednakost:

a) $\binom{12}{10} = \binom{12}{2}$

$$\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} = \frac{12 \cdot 11}{1 \cdot 2}$$

$$66 = 66$$

b) $\binom{13}{6} = \binom{13}{7}$

$$\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$$

$$1\,716 = 1\,716$$

$c)$ $$\binom{n+1}{n-1}+\binom{n+1}{n}=\binom{n+2}{n}$$

$$\frac{(n+1)!}{(n-1)!\,(n+1-n+1)!}+\frac{(n+1)!}{n!\,(n+1-n)!}=\frac{(n+2)!}{n!\,(n+2-n)!}$$

$$\frac{(n+1)!}{(n-1)!\,2!}+\frac{(n+1)!}{n!\,1!}=\frac{(n+2)!}{n!\,2!}$$

$$\frac{(n+1)n(n-1)!}{(n-1)!\cdot 1\cdot 2}+\frac{(n+1)n!}{n!}=\frac{(n+2)(n+1)n!}{n!\cdot 1\cdot 2}$$

$$\frac{(n+1)n}{2}+(n+1)=\frac{(n+2)(n+1)}{2}\quad /\cdot 2$$

$$n(n+1)+2(n+1)=(n+2)(n+1)$$

$$(n+1)(n+2)=(n+1)(n+2)$$

3. Odredi prirodan broj n tako da vrijedi jednakost:

a) $\binom{n}{5} = \binom{n}{3}$

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

$$\frac{(n-3)(n-4)}{20} = 1$$

$$(n-3)(n-4) = 20$$

$$n^2 - 4n - 3n + 12 - 20 = 0$$

$$n^2 - 7n - 8 = 0$$

$$n_{1,2} = \frac{7 \pm \sqrt{49+32}}{2} = \frac{7 \pm 9}{2} \Rightarrow n_1 = 8, n_2 = -1$$

Rješenje: $n = 8$

$b)\ \ 5\binom{n}{3} = \binom{n+2}{4}$

$$5\,\frac{n(n-1)(n-2)}{3!} = \frac{(n+2)(n+1)n(n-1)}{4!}$$

$$5\,\frac{(n-2)}{6} = \frac{(n+2)(n+1)}{24} \qquad /\cdot 24$$

$$5\cdot 4(n-2) = (n+2)(n+1)$$

$$20n - 40 = n^2 + n + 2n + 2$$

$$n^2 + 3n - 20n + 2 + 40 = 0$$

$$n^2 - 17n + 42 = 0$$

$$n_{1,2} = \frac{17 \pm \sqrt{289-168}}{2} = \frac{17 \pm 11}{2} \Rightarrow n_1 = 14, n_2 = 3$$